Exercise 1.2

Alter exercise 1.1 supposing instead that $f \gtrsim_T g$ iff

$\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$.

Which of our axioms (if any) does this preference relation violate?

Solution

Axiom 1.1A (Completeness). $f \gtrsim_s g$ or $g \gtrsim_s f$.

This preference relation does not violate Completeness. A lottery was defined (cf. p. 7) to be any function that specifies a nonnegative real number, $f(x \mid s)$, for every prize x in X and every state s in T, such that $\sum_{x \in X} f(x \mid s) = 1$ for every s in T. As a result of the definition, there must be at least a value of both $f(x \mid s)$ and $g(x \mid s)$ greater than zero, for every s in T . We can, thus, be sure that $\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\}$ and $\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$ are real numbers, and therefore that either $\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$ or $\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} \leq \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$.

This means, in terms of our decision maker's preferences, that either $f \gtrsim_T g$ or $g \gtrsim_T f$, Q.E.D.

Axiom 1.1B (Transitivity). If $f \gtrsim_s g$ and $g \gtrsim_s h$ then $f \gtrsim_s h$.

Transitivity is not violated. As we have just proved, $\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\}$ and $\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$ are both real numbers. It is therefore straightforward to check (cf. Transitivity proof in exercise 1.1) that it is always the case that if $f \gtrsim_s g$ and $g \gtrsim_s h$ then $f \gtrsim_s h$.

Axiom 1.3 (Monotonicity). If $f \succ_s g$ and $0 \leq \beta < \alpha \leq 1$, then $\alpha f + (1-\alpha) g \succ_s \beta f + (1-\beta) g$.

Monotonicity is violated. First, let us describe the antecedent and the consequent of the axiom in terms of our decision-maker's preference function. What the axiom asserts is:

If $\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} > \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$ and $0 \leq \beta < \alpha \leq 1$, then

$\sum_{s \in T} \min \{x \mid \alpha f(x \mid s) + (1-\alpha) g(x \mid s) > 0\} > \sum_{s \in T} \min \{x \mid \beta f(x \mid s) + (1-\beta) g(x \mid s) > 0\}$
.

Now we will provide a counterexample, in which the antecedent of the axiom is true and the consequent does not hold, in order to prove that Monotonicity is violated by this preference relation. Only afterwards will we explain why it is intuitive that the consequent does not need to hold, as we think it will be much more obvious by then.

Let us suppose that we have two possible prizes, with values 1 and 1000 (i.e., $X = \{1, 1000\}$) and only two possible states of the world (i.e.,$\mid T \mid = 2$). On the other hand we have two lotteries, f and g, such that $\forall s \in T$

$$f\left(1 \mid s\right) = 0$$

$$f\left(1000 \mid s\right) = 1$$

$$g\left(1 \mid s\right) = 1$$

$$g\left(1000 \mid s\right) = 0$$

That is, lottery f provides a prize $x = 1000$ and lottery g provides a prize $x = 1$, no matter which state of the world happens to occur.

For these lotteries, $\sum_{s \in T} \min\left\{x \mid f\left(x \mid s\right) > 0\right\} > \sum_{s \in T} \min\left\{x \mid g\left(x \mid s\right) > 0\right\}$, as $\sum_{s \in T} \min\left\{x \mid f\left(x \mid s\right) > 0\right\} = 2000$ and $\sum_{s \in T} \min\left\{x \mid g\left(x \mid s\right) > 0\right\} = 2$ (remember that there are only two possible states of the world, so we only have two prizes to add for each lottery).

Now, which is the value of $\sum_{s \in T} \min\left\{x \mid \alpha f\left(x \mid s\right) + \left(1 - \alpha\right) g\left(x \mid s\right) > 0\right\}$ and $\sum_{s \in T} \min\left\{x \mid \beta f\left(x \mid s\right) + \left(1 - \beta\right) g\left(x \mid s\right) > 0\right\}$, considering of course that $0 \leq \beta < \alpha \leq 1$?

Let us see: $\forall s \in T$,

$\alpha f\left(1 \mid s\right) + \left(1 - \alpha\right) g\left(1 \mid s\right) = 1 - \alpha$ and $\alpha f\left(1000 \mid s\right) + \left(1 - \alpha\right) g\left(1000 \mid s\right) = \alpha$, which means that for both prizes this linear combination of probabilities is above zero if $\alpha \neq 1$ and $\alpha \neq 0$. That is, for values of α different from 1 the minimum value of $x \mid \alpha f\left(x \mid s\right) + \left(1 - \alpha\right) g\left(x \mid s\right) > 0$ is $x = 1$ (if $\alpha = 0$ only the first of the equalities holds the condition of being above zero, and the minimum prize under that condition is obviously still $x = 1$). In conclusion:

If $\alpha \neq 1$, $\sum_{s \in T} \min\left\{x \mid \alpha f\left(x \mid s\right) + \left(1 - \alpha\right) g\left(x \mid s\right) > 0\right\}$=2

Using exactly the same reasoning to check the possible values of $\beta f\left(x \mid s\right) + \left(1 - \beta\right) g\left(x \mid s\right)$and the minimum prizes when these values are over zero, we have that

$\forall s \in T$, $\beta f\left(1 \mid s\right) + \left(1 - \beta\right) g\left(1 \mid s\right) = 1 - \beta$ and $\beta f\left(1000 \mid s\right) + \left(1 - \beta\right) g\left(x1000 \mid s\right) = \beta$. So:

If $\beta \neq 1$ then $\sum_{s \in T} \min\left\{x \mid \beta f\left(x \mid s\right) + \left(1 - \beta\right) g\left(x \mid s\right) > 0\right\} = 2$

Given that $0 \leq \beta < \alpha \leq 1$, then $\sum_{s \in T} \min\left\{x \mid \beta f\left(x \mid s\right) + \left(1 - \beta\right) g\left(x \mid s\right) > 0\right\} = 2$ for every possible β.

To sum up, if $\alpha \neq 1$, the consequent of the axiom does not hold, because both sides of the inequality have the same value, namely 2. Of course, $2 \not> 2$, so $\sum_{s \in T} \min\left\{x \mid \alpha f\left(x \mid s\right) + \left(1 - \alpha\right) g\left(x \mid s\right) > 0\right\} \not> \sum_{s \in T} \min\left\{x \mid \beta f\left(x \mid s\right) + \left(1 - \beta\right) g\left(x \mid s\right) > 0\right\}$, these lotteries yield a counterexample to the axiom for every α and β such that $0 \leq \beta < \alpha \leq 1$.

Again (as we already saw in Exercise 1.1) proving that the axiom does not hold leads to the conclusion that the preference function under consideration does not grow monotonically when α grows. Actually,as we have just seen, the value of the function is always 2 except when α is 1, when its value turns into 1000. Never does it grow, except in one only point.

Why does this happen? When comparing minimum prizes for f and g, our decision-maker leaves out, for each possible state, those impossible to obtain with each lottery. Nevertheless, when comparing prizes with a probability of being obtained based on a linear combination of both lotteries, prizes that were left out in his first choice can appear as obtainable.

Axiom 1.4 (Continuity) . If $f \gtrsim_s g$ and $g \gtrsim_s h$, then there exists some number γ such that $0 \leq \gamma \leq 1$ and $g \sim_s \gamma f + (1-\gamma) h$.

Continuity is violated. In terms of our decision-maker in particular, the axiom asserts that if

$$\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid h(x \mid s) > 0\}$$

then there exists some number γ such that $0 \leq \gamma \leq 1$ such that

$$\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} = \sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\},$$

which is not true in all cases. Let us see one of the cases in which the axiom fails:

Suppose we have two possible states of the world ($\mid T \mid = 2$), three different possible prizes for each state ($x = \{1, 2, 3\}$) and three lotteries f, g and h, such that $\forall s \in T$

$f(1 \mid s) = 0$, $f(2 \mid s) = 0$ and $f(3 \mid s) = 1$;
$g(1 \mid s) = 0$, $g(2 \mid s) = 1$and $g(3 \mid s) = 0$;
$h(1 \mid s) = 1$, $h(2 \mid s) = 0$ and $h(3 \mid s) = 0$.

That is, lottery f always gives a prize with a value $x = 3$, lottery g always gives a prize $x = 2$and lottery h always gives a prize with a value $x = 1$. The actual state of the world is irrelevant, as probabilities do not change with such states.

Let us check, first of all, that the antecedent of the axiom is true in this particular case:

$\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} = 6$ (prize is $x = 3$ for each of the two possible states)
$\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} = 4$
$\sum_{s \in T} \min \{x \mid h(x \mid s) > 0\} = 2.$
Now, $6 > 4 > 2$, so it is true that
$\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid h(x \mid s) > 0\}$.

Now let us see if there exists some number γ such that $0 \leq \gamma \leq 1$ and $\sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\} = 4$, as 4 is the value of $\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\}$in our example.

If $\gamma = 1$, then $\sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\} = 6$, as $\sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\}$ becomes$\sum_{s \in T} \min \{x \mid f(x \mid s) > 0\}$.

If $\gamma = 0$, then $\sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\} = 2$, as it turns into $\sum_{s \in T} \min \{x \mid h(x \mid s) > 0\}$.

In fact, $\forall \gamma \neq 1$, $\sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\} = 2$, as prizes that are obtainable in this terms -those for which $(\gamma f + (1-\gamma) h) > 0$- are $x = 1$ and $x = 3$, being $x = 1$the minimum.

That is, there does not exist any number γ such that $0 \leq \gamma \leq 1$and $\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} = \sum_{s \in T} \min \{x \mid (\gamma f + (1-\gamma) h) > 0\}$. The axiom does not hold for this particular example, which proves that it is violated by the defined preference function. Actually, if we consider the preference relation as a function of γ, a discontinuity clearly appears in $\gamma = 1$.

The reason why the consequent of the axiom does not need to hold -the antecedent being true- when using this kind of function is that it simply does not ensure that prizes that are obtainable with lottery g -and so candidates for being chosen as minimum- are obtainable at all with a linear combination of lotteries f and h, that is, that probability given by such a linear combination will be over zero.

Axiom 1.5A (Objective Substitution) If $e \gtrsim_s f$ and $g \gtrsim_s h$ and $0 \leq \alpha \leq 1$, then $\alpha e + (1-\alpha) g \gtrsim_s \alpha f + (1-\alpha) h$.

Objective substitution is also violated. According to both the antecedent of the axiom and our decision-maker's preference relation, we know that

$\sum_{s \in T} \min \{x \mid e(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid f(x \mid s) > 0\}$ and $\sum_{s \in T} \min \{x \mid g(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid h(x \mid s) > 0\}$.

Now, in order to prove that the consequent does not hold for every possible case, we must prove that it is not true that $\forall \alpha \sum_{s \in T} \min \{x \mid \alpha e(x \mid s) + (1-\alpha) g(x \mid s) > 0\} \geq \sum_{s \in T} \min \{x \mid \alpha f(x \mid s) + (1-\alpha) h(x \mid s) > 0\}$.

Checking that it actually holds with values $\alpha = 0$and $\alpha = 1$ is straigtforward and intuitive enough, as it makes reference to those cases when probabilities that must be above zero -for minimum prizes to be picked- in both sides of the inequality are simply probabilities that correspond to lotteries as they were defined, that is, to either e and f or g and h.

Now we have to check what happens with the values of $\sum_{s \in T} \min \{x \mid \alpha e(x \mid s) + (1-\alpha) g(x \mid s) > 0\}$ and $\sum_{s \in T} \min \{x \mid \alpha f(x \mid s) + (1-\alpha) h(x \mid s) > 0\}$when $0 < \alpha < 1$. First of all let us explain why it would not hold, from an intuitive point of view . The reason is analogous to the one that explained why Monotonicity and Continuity were violated: when we calculate minimum prizes for each state and lottery -minimum prizes that, once properly added, assure that the antecedent conditions hold- there could be even lesser prizes for one or many of those states that are not considered because their probability of being won is zero; nevertheless, when calculating minimum prizes such that linear combinations of their probabilities are over zero, those lesser prizes might this time have to be considered, with, in case that lotteries are defined in a concrete way, could lead to values of $\sum_{s\in T}\min\left\{x \mid \alpha e\left(x\mid s\right)+\left(1-\alpha\right)g\left(x\mid s\right)>0\right\}$ and $\sum_{s\in T}\min\left\{x \mid \alpha f\left(x\mid s\right)+\left(1-\alpha\right)h\left(x\mid s\right)>0\right\}$whose relation is the opposite to the one we are looking for. To prove this, let us provide a counterexample:

Suppose we have two states of the world, $T=\{s_1,s_2\}$ and any set of prizes X. Suppose that lotteries e,f, g, and h are defined as it follows:

$e\left(x\mid s_1\right)=[0,5]\left(x\mid s_1\right)$ (remember that $[i]\left(x\mid t\right)$ is a lottery such that $[i]\left(i\mid t\right)=1$and $[i]\left(j\mid t\right)=0$, if $j\neq i$)

$e\left(x\mid s_2\right)=[1]\left(x\mid s_2\right)$

$g\left(x\mid s_1\right)=[1]\left(x\mid s_1\right)$

$g\left(x\mid s_2\right)=[0,5]\left(x\mid s_2\right)$

$f\left(x\mid s_i\right)=[0,7]\left(x\mid s_i\right)=h\left(x\mid s_i\right)$

Let us check, first of all, that conditions under which the axiom is built actually hold:

$\sum_{s\in T}\min\left\{x\mid e\left(x\mid s\right)>0\right\}=1,5$

$\sum_{s\in T}\min\left\{x\mid f\left(x\mid s\right)>0\right\}=1,4$

$\sum_{s\in T}\min\left\{x\mid g\left(x\mid s\right)>0\right\}=1,5$

$\sum_{s\in T}\min\left\{x\mid h\left(x\mid s\right)>0\right\}=1,4$

That is, $\sum_{s\in T}\min\left\{x\mid e\left(x\mid s\right)>0\right\}\geq\sum_{s\in T}\min\left\{x\mid f\left(x\mid s\right)>0\right\}$and $\sum_{s\in T}\min\left\{x\mid g\left(x\mid s\right)>0\right\}\geq\sum_{s\in T}\min\left\{x\mid h\left(x\mid s\right)>0\right\}$.

Now let us calculate the values of the $\sum$ that constitutes the consequent of the axiom. $\forall\alpha$ such that $0<\alpha<1$:

$\sum_{s\in T}\min\left\{x\mid \alpha e\left(x\mid s\right)+\left(1-\alpha\right)g\left(x\mid s\right)>0\right\}=1$

$\sum_{s\in T}\min\left\{x\mid \alpha f\left(x\mid s\right)+\left(1-\alpha\right)h\left(x\mid s\right)>0\right\}=1,4$

That is, in this case $\sum_{s\in T}\min\left\{x\mid \alpha e\left(x\mid s\right)+\left(1-\alpha\right)g\left(x\mid s\right)>0\right\}\leq\sum_{s\in T}\min\{x\mid \alpha f\left(x\mid s\right)+\left(1-\alpha\right)h\,($ and so the axiom does not hold.

Axiom 1.5B (Strict Objective Substitution) If $e\succ_s f$ and $g\gtrsim_s h$ and $0<\alpha\leq 1$, then $\alpha e+\left(1-\alpha\right)g\succ_s\alpha f+\left(1-\alpha\right)h$.

The same counterexample is valid to prove that Strict Objective Substitution is also violated by our decision-maker's preferences, as relations defined in it are such that $e\succ_s f$,$g\gtrsim_s h$, $0<\alpha\leq 1$and $\alpha e+\left(1-\alpha\right)g\succ_s\alpha f+\left(1-\alpha\right)h$.

Axiom 1.6A (Subjective Substitution) If $f\gtrsim_s g$ and $f\gtrsim_T g$ and $S\cap T=0$, then $f\gtrsim_{s\cup T} g$.

This axiom is not violated. What we know, in terms of our decison-maker's preferences, is that $\sum_{s\in S}\min\left\{x\mid f\left(x\mid s\right)>0\right\}\geq\sum_{s\in S}\min\left\{x\mid g\left(x\mid s\right)>0\right\}$ and that $\sum_{s\in T}\min\left\{x\mid f\left(x\mid s\right)>0\right\}\geq\sum_{s\in T}\min\left\{x\mid g\left(x\mid s\right)>0\right\}$.

By adding both inequalities -and considering when doing it that $S\cap T=0$- we have that

$\sum_{s\in S\cup T}\min\{x \mid f(x \mid s)>0\} \geq \sum_{s\in S\cup T}\min\{x \mid g(x \mid s)>0\}$, which constitutes exactly the consequent of the axiom.

Axiom **1.6B (Strict Subjective Substitution)** If $f \succ_s g$ and $f \succ_T g$ and $S\cap T=0$, then $f \succ_{s\cup T} g$.

It can be proved in the same way.

Axiom 1.7 (Interest) For every state t in Ω,there exist prizes y and z in X such that $[y] \succ_{\{t\}} [z]$.

As it happened in exercise 1.1, the axiom is not violated if X has at least two members.

Axiom 1.8 (State Neutrality) For any two states r and t in Ω,if $f(\cdot \mid t)=f(\cdot \mid r)$ and $g(\cdot \mid t)=fg(\cdot \mid r)$ and $f \gtrsim_{\{r\}} g$, then $f \gtrsim_{\{t\}} g$.

Again as it happened in exercise 1.1, for State Neutrality to be violated the contrary would have been to be explicitly stated when describing the problem.