

Chapter 1

Exercises

1. Suppose that the set of prizes X is a finite subset of \mathbf{R} , the set of real numbers, and a prize x denotes an award of x dollars. A decision-maker says that, if he knew that the true state of the world was in some set T , then he would weakly prefer a lottery f over another lottery g (that is, $f \succeq_T g$)

$$\min_{s \in T} \sum_{x \in X} x f(x | s) \geq \min_{s \in T} \sum_{x \in X} x g(x | s)$$

(that is, he prefers the lottery that gives the higher expected payoff in the worst possible state.) Which of our axioms (if any) does this preference relation violate?

Solution to Exercise 1

Axiom 1.1A (COMPLETENESS). $f \succeq_T g$ or $g \succeq_T f$

The preference function of our decision-maker does not violate axiom 1.1.A. We know that any lottery f must specify a nonnegative real number $f(x | t)$, for every price x in X and every state t in Ω (cf. p. 7) and, given that every member of X is a real number, then

$\forall s \in T, \forall x \in X \sum_{x \in X} x f(x | s)$ is a real number. Therefore

$\min_{s \in T} \sum_{x \in X} x f(x | s)$ exists for every lottery f , and is a real number. Now, our decision-maker will prefer lottery f to g iff

$\min_{s \in T} \sum_{x \in X} x f(x | s) \geq \min_{s \in T} \sum_{x \in X} x g(x | s)$, and will prefer lottery g to f iff

$\min_{s \in T} \sum_{x \in X} x g(x | s) \geq \min_{s \in T} \sum_{x \in X} x f(x | s)$. As we have just seen, these are just two real numbers, and for every pair of real numbers a and b , either $a \geq b$ or $b \geq a$. So, for him, either $f \succeq_T g$ or $g \succeq_T f$ Q.E.D.

Axiom 1.1B (TRANSITIVITY). If $f \succeq_T g$ and $g \succeq_T h$ then $f \succeq_T h$

Axiom 1.1B is not violated. Let us suppose that $f \succeq_T g$ and $g \succeq_T h$ for our decision-maker. We will show that, then, $f \succeq_T h$ for him too. According to the hypothesis we have just made,

$$\min_{s \in T} \sum_{x \in X} x f(x | s) \geq \min_{s \in T} \sum_{x \in X} x g(x | s) \text{ and}$$

$$\min_{s \in T} \sum_{x \in X} x g(x | s) \geq \min_{s \in T} \sum_{x \in X} x h(x | s).$$

Now, as we have seen in the proof of axiom 1.1A, the three relata in the two inequalities above are real numbers. But, for every real numbers a , b and c , if $a \geq b$ and $b \geq c$ then $a \geq c$; so, in this particular case,

$$\min_{s \in T} \sum_{x \in X} x f(x | s) \geq \min_{s \in T} \sum_{x \in X} x h(x | s). \text{ Q.E.D.}$$

Axiom 1.2 (RELEVANCE). If $f(\cdot | t) = g(\cdot | t) \quad \forall t \in S$, then $f \sim_S g$.

Axiom 1.2 is not violated. If the value of $f(x | t)$ is equal to the value of $g(x | t)$ for every x in X and every t in S then, obviously,

$\min_{s \in T} \sum_{x \in X} x f(x | s) = \min_{s \in T} \sum_{x \in X} x g(x | s)$. That is, for our decision-maker, $f \sim_S g$. Q.E.D.

Axiom 1.3 (MONOTONICITY). If $f >_S h$ and $0 \leq \beta < \alpha \leq 1$, then $\alpha f + (1 - \alpha)g >_S \beta f + (1 - \beta)g$.

This axiom is violated by the preference function of our decision-maker. First, we will suppose that the antecedent of the axiom is true and show why the consequent does not need to hold. Then we will provide a counterexample to the axiom.

We will suppose that $f >_S h$, that is, that $\min_{s \in T} \sum_{x \in X} x f(x | s) > \min_{s \in T} \sum_{x \in X} x h(x | s)$. If the axiom is true, then, for every two real numbers α and β such that $0 \leq \beta < \alpha \leq 1$, it must be true that $\alpha f + (1 - \alpha)g >_S \beta f + (1 - \beta)g$. That is, (according to the preferences of our decision-maker) the following inequality must hold:

$$\text{(INEQ1)} \quad \min_{s \in T} \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] > \min_{s \in T} \sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)].$$

What we do have, instead, (after some straightforward manipulation of the antecedent of the axiom) is:

$$\text{(INEQ2)} \quad \alpha \min_{s \in T} \sum_{x \in X} x f(x | s) + (1 - \alpha) \min_{s \in T} \sum_{x \in X} x h(x | s) > \beta \min_{s \in T} \sum_{x \in X} x f(x | s) + (1 - \beta) \min_{s \in T} \sum_{x \in X} x h(x | s)$$

Can we massage INEQ2 so that we turn it into INEQ1? Well, we do know that

$$\text{(INEQ3)} \quad \min_{s \in T} \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] \geq \alpha \min_{s \in T} \sum_{x \in X} x f(x | s) + (1 - \alpha) \min_{s \in T} \sum_{x \in X} x h(x | s)$$

This is because, in the left-hand side (LHS) of this inequality, we have to keep the state s constant when calculating the minimum of a linear combination of lotteries f and h , while, in the right-hand side (RHS), we can choose a different state for the minimum of each lottery. If the state s that makes $\sum_{x \in X} xf(x | s)$ minimum is the same state r that makes $\sum_{x \in X} xg(x | r)$ minimum, then LHS and RHS are equal. If r and s are different, though, LHS will be bigger than RHS. Now, if we also had

$$\text{(INEQ4)} \quad \min_{s \in T} \sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)] \leq \beta \min_{s \in T} \sum_{x \in X} xf(x | s) + (1 - \beta) \min_{s \in T} \sum_{x \in X} h(x | s)$$

we could pile up INEQ3, INEQ2 and then INEQ4 to get INEQ1. But the problem, of course, is that INEQ4 is false. Actually LHS and RHS are in the opposite relation, for exactly the same reasons that we have mentioned when talking about INEQ3. Given the inequations that we do know to be true, it is not possible to ensure that the consequent of the axiom will hold.

The easiest way to prove that, in fact, the axiom does *not* hold is to provide a counterexample. Consider the following example. We have $\Omega = \{a, b, c\}$ (i.e., there are only three possible states of the world) and lotteries f and h such that:

$$\begin{aligned} f(x | s) &= [3](x | s) \text{ (i.e., the lottery that always gives a prize of 3 dollars),} \\ &\text{when } s = b \\ f(x | s) &= [4](x | s) \text{ when } s \neq b \text{ (that is, } s = a \text{ or } s = c) \end{aligned}$$

$$\begin{aligned} h(x | s) &= [2](x | s) \text{ (i.e., the lottery that always gives a prize of 3 dollars),} \\ &\text{when } s = a \\ h(x | s) &= [10](x | s) \text{ when } s \neq a \text{ (that is, } s = b \text{ or } s = c) \end{aligned}$$

Now, for these lotteries, $\min_{s \in T} \sum_{x \in X} xf(x | s) > \min_{s \in T} \sum_{x \in X} xh(x | s)$, because $3 > 2$, so INEQ1

$$\text{(INEQ1)} \quad \min_{s \in T} \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] > \min_{s \in T} \sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)]$$

should be true for every value of real numbers α and β such that $0 \leq \beta < \alpha \leq 1$. But it is not, if we take $\alpha = 0,95$ and $\beta = 0,9$. Let us calculate it. What is the value of the LHS of INEQ1, that is, of $\min_{s \in T} \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)]$? Let us calculate the \sum for all three states.

$$\begin{aligned} \text{When } s = a, \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] &= 0,95 \cdot 4 + 0,05 \cdot 2 = 3,9 \\ \text{When } s = b, \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] &= 0,95 \cdot 3 + 0,05 \cdot 10 = 3,35 \\ \text{When } s = c, \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)] &= 0,95 \cdot 4 + 0,05 \cdot 10 = 4,3 \end{aligned}$$

The minimum for the LHS happens when $s = b$ and is equal to 3,35. Now let us repeat the operation with the RHS:

$$\begin{aligned} \text{When } s = a, \sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)] &= 0,9 \cdot 4 + 0,1 \cdot 2 = 3,8 \\ \text{When } s = b, \sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)] &= 0,9 \cdot 3 + 0,1 \cdot 10 = 3,7 \end{aligned}$$

When $s = c$, $\sum_{x \in X} x [\beta f(x | s) + (1 - \beta)h(x | s)] = 0,9 \cdot 4 + 0,1 \cdot 10 = 4,6$

The minimum for the RHS, again, happens when $s = b$ and is equal to 3,7. That is, contrary to what the axiom needs, the RHS is bigger than the LHS. The axiom is violated.

The thing is, for the axiom to hold, the preference function (in this case, $\min_{s \in T} \sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)]$) must grow *monotonically* (hence the name of the axiom) when α grows. To see what is happening with our example, let us depict the following in a graph (Figure 1) for all three possible states a , b and c .

y-axis: $\sum_{x \in X} x [\alpha f(x | s) + (1 - \alpha)h(x | s)]$

x-axis: α

As we can see, the minimum of all three curves (the bottom line formed by the lowest curve at each point) grows steadily up until a value of α slightly bigger than 0,85; after that, it decreases. The preference function, thus, does not grow monotonically and the axiom is violated.

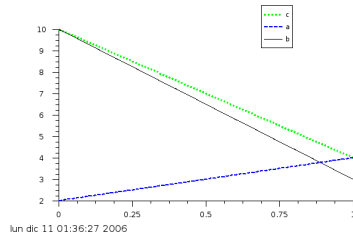


Figure 1: A non-monotonically growing preference function

Axiom 1.4 (CONTINUITY). If $f \geq_S g$ and $g \geq_S h$, then there exists some number γ such that $0 \leq \gamma \leq 1$ and $g \sim_S \gamma f + (1 - \gamma)h$.

This axiom is not violated. To see this, we will show a way of constructing γ for any lotteries f , g and h such that $f \geq_S g$ and $g \geq_S h$. Let me introduce some abbreviations first:

- $\min f$ for $\min_{s \in T} \sum_{x \in X} x f(x | s)$
- $f(s)$ for $\sum_{x \in X} x f(x | s)$

Given the preference function of our decision maker, $\min f \geq \min g \geq \min h$. We want to find a γ such that

$$\min g = \min [\gamma f + (1 - \gamma)h].$$

Let $a \in T$ be the state such that $h(a) = \min h$. Now, obviously, $h(a) < \min g$. Also, given that $\min f > \min g$, $f(a) > \min g$. Therefore, there is a linear combination of $f(a)$ and $h(a)$ that is equal to $\min g$. That is,

$$\exists \delta [\delta f(a) + (1 - \delta)h(a)] = \min g.$$

Now there are two possibilities:

1. $\delta f(a) + (1 - \delta)h(a) = \min [\delta f + (1 - \delta)h]$. If this is true, then $\gamma = \delta$ and we have finished.
2. $\delta f(a) + (1 - \delta)h(a) \neq \min [\delta f + (1 - \delta)h]$.

If 2 is true, this means that

$$\text{(INEQ5)} \quad \exists b \in T [\delta f(b) + (1 - \delta)h(b)] < [\delta f(a) + (1 - \delta)h(a)].$$

Now, $h(b) \geq h(a) = \min h$, and $f(b) \leq f(a)$. Otherwise INEQ5 won't hold. Besides $f(b) \geq \min g$ because every $f(x)$ is. So, again, there is a linear combination of $f(b)$ and $h(b)$ that is equal to $\min g$:

$$\exists \delta' [\delta' f(b) + (1 - \delta')h(b)] = \min g.$$

Given that $\delta' f(b) + (1 - \delta')h(b) = \min g > \delta f(b) + (1 - \delta)h(b)$, we know that $\delta' > \delta$. And again:

1. $\delta' f(b) + (1 - \delta')h(b) = \min [\delta f + (1 - \delta')h]$. If this is true, then $\gamma = \delta'$ and we have finished.
2. ...

So, we have a process in which we find a sequence of $\delta^{[n]}$, such that for each such $\delta^{[n]}$, either it is the γ we were looking for, or there is a $\delta^{[n+1]}$. This process cannot go on forever, because $\delta^{[n]} \leq 1$. If, actually, $\delta^{[n]} = 1$, this means $\min f = \min g = \min h$. Q.E.D.

Axiom 1.5A (OBJECTIVE SUBSTITUTION). If $e \geq_S f$ and $g \geq_S h$ and $0 \leq \alpha \leq 1$ then $\alpha e + (1 - \alpha)g \geq_S \alpha f + (1 - \alpha)h$.

This axiom is violated. Let's see a counter-example. We have $\Omega = \{a, b, c, d, i\}$ (i.e., there are only five possible states of the world) and lotteries e, f, g and h such that:

$e(x | s) = [11](x | s)$ (i.e., the lottery that always gives a prize of 10 dollars), when $s = c$

$e(x | s) = [12](x | s)$ when $s \neq c$ (that is, $s = a$ or $s = b$ or $s = d$ or $s = i$)

$f(x | s) = [10](x | s)$ when $s = a$

$f(x | s) = [20](x | s)$ when $s \neq a$

$g(x | s) = [11](x | s)$ when $s = d$

$g(x | s) = [12](x | s)$ when $s \neq d$

$h(x | s) = [10](x | s)$ when $s = b$

$h(x | s) = [20](x | s)$ when $s \neq b$

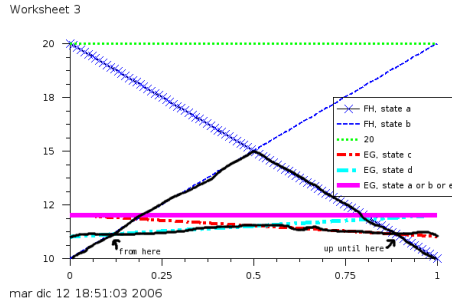


Figure 2

For these choices of lottery, it is true that $\min e \geq \min f$ and $\min g \geq \min h$ so, if the preference function of our decision-maker is to comply with axiom 1.5A, it must also be true that $\forall \alpha \quad \min(\alpha e + (1 - \alpha)g) \geq_S \min(\alpha f + (1 - \alpha)h)$. Let us abbreviate:

- $EG \equiv \alpha e + (1 - \alpha)g$
- $FH \equiv \alpha f + (1 - \alpha)h$

Now, let us calculate EG and FH for all five states:

- When $s = a$, $FH = 10\alpha + 20(1 - \alpha)$
- When $s = b$, $FH = 20\alpha + 10(1 - \alpha)$
- When $s \neq a, s \neq b$, $FH = 20$
- When $s = c$, $EG = 11\alpha + 12(1 - \alpha)$
- When $s = d$, $EG = 12\alpha + 11(1 - \alpha)$
- When $s \neq c, s \neq d$, $EG = 12$

In the graph (Figure 2), I have marked the minima for both the EG and the FH curves. It is easily seen that, for a big range of values of α , $\min FH > \min EG$. All those values provide counterexamples to the axiom.

Axiom 1.5B (STRICT OBJECTIVE SUBSTITUTION). If $e >_S f$ and $g >_S h$ and $0 \leq \alpha \leq 1$ then $\alpha e + (1 - \alpha)g >_S \alpha f + (1 - \alpha)h$.

The very same counterexample works against this axiom too. It is only that now the crossing-points between the minima curves also count as counterexamples against the axiom -unlike in 1.5A.

Axiom 1.6A (SUBJECTIVE SUBSTITUTION). If $f \geq_S g$ and $f \geq_T g$ and $S \cap T = \emptyset$, then $f \geq_{S \cup T} g$.

Our decision-makers preference function complies with this axiom. We know that

$$\min_{s \in S} \sum_{x \in X} x f(x | s) \geq \min_{s \in S} \sum_{x \in X} x g(x | s), \text{ and}$$

$$\min_{s \in T} \sum_{x \in X} x f(x | s) \geq \min_{s \in T} \sum_{x \in X} x g(x | s). \text{ Obviously,}$$

$\min_{s \in S \cup T} \sum_{x \in X} x f(x | s) = \min [\min_{s \in S} \sum_{x \in X} x f(x | s), \min_{s \in T} \sum_{x \in X} x f(x | s)]$. And the same applies for g . It is straightforward to see that, if all of this is true, then

$$\min_{s \in S \cup T} \sum_{x \in X} x f(x | s) \geq \min_{s \in S \cup T} \sum_{x \in X} x g(x | s). \text{ Q.E.D.}$$

Axiom 1.6B (SUBJECTIVE SUBSTITUTION). If $f >_S g$ and $f >_T g$ and $S \cap T = \emptyset$, then $f >_{S \cup T} g$.

It can be proved in the same way.

Axioms 1.7 is not violated if the prize set has at least two members. For axiom 1.8 to be violated, it should have been explicitly denied in the description of the problem.